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THE PERIODIC MOTIONS OF A GAS†

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Two-dimensional periodic motions of a gas, described by a class of rotational and rotational–symmetrical exact solutions of the gas-dynamic equations, are considered. The investigation is based on constructing first integrals and a lemma on the existence of periodic functions, defined by quadratures of special form. The idea of limiting relations is introduced, which enable approximate relations to be established between the constituent parameters and which give a qualitative representation on the form of the periodic gas motion being investigated. In addition to examples of a limit analysis of previously known motions, an existence theorem of a new form of periodic motion called a "gaseous pinion" is presented. © 2001 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The time-periodicity of the motion of an ideal gas when there are no external forces distributed over the mass is an exceptional phenomenon. In view of the finiteness of the propagation velocity of perturbations in a gas, this form of motion can easily collapse. From this point of view, time-periodic motions are similar to steady gas flows. Hence, the question of the existence of this form of motion is not trivial. Only two examples of such motion are known so far: rest and rigid-body rotation. This paper proves the existence of new forms of gas motions, that are periodic in time. They occur in classes of two-dimensional rotational and rotational-symmetrical exact solutions of the gas-dynamic equations.

Two-dimensional motions of a polytropic gas with an equation of state (γ is the adiabatic exponent)

$$p = s\rho^{\gamma} \tag{1.1}$$

are described by the required quantities: the radial component of the velocity vector V and the circular component of the velocity vector W, the density ρ, the pressure p and the entropy s, considered as a function of the time t and the polar coordinates r, θ. The corresponding system of differential equations admits of (in Lie's sense) a two-dimensional group $G_2(H_1, H_2)$ with a basis of Lie algebra of operators

$$H_1 = \alpha\partial_t - \beta\partial_\theta \quad (\alpha^2 + \beta^2 \neq 0)$$
$$H_2 = r\partial_r + V\partial_V + W\partial_W + n\rho\partial_\rho + (n+2)\,p\partial_p + ms\partial_s$$

where α, β and n are arbitrary constants and

$$m = n + 2 - n\gamma \tag{1.2}$$

The group G_2 generates an invariant submodel – a system of ordinary differential equations for certain functions of the independent variable

$$\xi = \alpha\theta + \beta t \tag{1.3}$$

Since the group G_2 is Abelian, the G_2-submodel can be constructed in two stages: initially as an invariant $G_1(H_1)$-submodel and then one can obtain from it an invariant $G_1(H_2)$-submodel [1]. This method is convenient here since it simplifies the integration of the equations of the G_2-submodel.

At the first stage, all the required quantities V, W, ρ, p and s are regarded as functions of the independent variables (ξ, r), while the equations of the $G_1(H_1)$-submodel have the form

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$$
\begin{aligned}
&DV+\rho^{-1}rp_r=W^2\\
&DW+\rho^{-1}\alpha p_\xi=-VW\\
&D\rho+\rho(V+rV_r+\alpha W_\xi)=0\\
&Ds=0
\end{aligned}
\tag{1.4}
$$

with total derivative operator $D = (\alpha W + \beta r)\,\partial_\xi + rV\partial_r$ and equation of state (1.1).

At the second stage, in system (1.4) expressions for the required quantities are established in terms of the invariants $A, \dots, S$ of the operator H_2

$$
V=rA,\quad W=rB,\quad \rho=r^nR,\quad p=r^{n+2}P,\quad s=r^mS \tag{1.5}
$$

where $A, \dots, S$ are functions of ξ (1.3) only. This leads to the following equations of the G_2-submodel (the prime denotes derivatives with respect to ξ)

$$
\begin{aligned}
&(\alpha B+\beta)A'+A^2-B^2+(n+2)P/R=0\\
&(\alpha B+\beta)B'+2AB+\alpha P'/R=0\\
&(\alpha B+\beta)R'+(n+2)AR+\alpha RB'=0\\
&(\alpha B+\beta)S'+mAS=0,\quad P=SR^\gamma
\end{aligned}
\tag{1.6}
$$

Remark 1. The parameters α and β are not essential and are only introduced to make the writing of systems (1.4) and (1.6) uniform. They can be changed by multiplying the operator H_1 by an arbitrary factor and by stretching the time t. These changes lead to subgroups G_2', contained within G_2, and thereby lead to solutions which can be obtained from the "standard" ones by a change of variables. Hence, it is sufficient to consider solely classes of "standard" solutions, for which $(\alpha, \beta) = (0, 1)$ or $(\alpha, \beta) = (1, \beta)$, where $\beta = 0$ or $\beta = 1$.

2. PERIODICITY CONDITIONS

The problem consists of finding solutions of system (1.6) which describe gas motions that are periodic in t. In this section we will derive the conditions for such solutions to exist. Here we will assume that the circular velocity $W > 0$ (this property is satisfied in all the examples considered below).

The equation of the trajectories of the gas particles $dr/dt = V$, $r\,d\theta/dt = W$ for solutions of the form (1.5) reduce to the following

$$
dr/dt=rA(\xi),\quad d\theta/dt=B(\xi) \tag{2.1}
$$

while the evolution of ξ (1.3) along the trajectories is described by the equation

$$
d\xi/dt=\alpha B(\xi)+\beta \tag{2.2}
$$

Suppose the functions $r(t)$ and $\theta(t)$ form a solution of system (2.1), i.e. describe the orbits of the motion of the gas particles. For periodicity it is necessary that the orbits should be closed curves in the (r, θ) plane. Suppose T is the minimum orbital period of such motion, i.e. for any t the following equations hold

$$
r(t+T)=r(t),\quad \theta(t+T)=\theta(t)+2\pi \tag{2.3}
$$

In this case the invariant coordinate ξ (1.3) obtains an increment

$$
\xi(t+T)=\xi(t)+\Phi_1,\quad \Phi_1=2\pi\alpha+\beta T \tag{2.4}
$$

If follows from (2.1) that the functions $A(\xi)$ and $B(\xi)$ must be periodic with period Φ_1, which, generally speaking, will not necessarily be the minimum period of these functions. If the latter is Φ, we must have $\Phi_1 = \nu\Phi$ with a certain integer $\nu \neq 0$. The quantity Φ will be called the phase period. Thus, the orbital and phase periods are related by the expression

$$\nu\Phi = 2\pi\alpha + \beta T \tag{2.5}$$

Nevertheless, it is obvious that the first relation of (2.4), by virtue of Eq. (2.2) with right-hand side that is positive and Φ-periodic in ξ, is equivalent to

$$T = \nu\int_0^\Phi [\alpha B(\xi) + \beta]^{-1} d\xi \tag{2.6}$$

From (2.1) and (2.2) we can determine invariant trajectories with the equation $r = r(\xi)$, which will be Φ-periodic in the plane of the polar coordinates (r, ξ). For the solution (1.5) to be periodic it is necessary that the invariant trajectories should be closed curves, i.e. that $r(\xi + 2\pi) = r(\xi)$. This will be true if and only if the following relation is satisfied with a certain integer $\mu \neq 0$

$$\mu\Phi = 2\pi \tag{2.7}$$

Remark 2. If $\alpha = 0$, Eqs (2.5) and (2.6) yield the same relation

$$\nu\Phi = \beta T \tag{2.8}$$

In this case $\xi = \beta t$ and the period Φ can be calculated in terms of a quadrature of the second equation of (2.1). By virtue of relation (2.8) the condition that $\theta(t)$ should change over the period T by 2π reduces to the equation

$$\nu\int_0^\Phi B(\xi) d\xi = 2\pi\beta$$

We finally obtain the following result.

Proposition 1. In order for Eqs (1.5) to describe the time-periodic gas motion with orbital period T, it is necessary and sufficient that the corresponding solution of system (1.6) should be periodic with phase period Φ, where the periods T and Φ are related with certain non-zero integers μ and ν by relations (2.4)–(2.7), namely

$$T = \nu\int_0^\Phi [\alpha B(\xi) + \beta]^{-1} d\xi, \quad \nu\Phi = 2\pi\alpha + \beta T, \quad \mu\Phi = 2\pi \tag{2.9}$$

the first of which, when $\alpha = 0$, is replaced by the relation

$$\nu\int_0^\Phi B(\xi) d\xi = 2\pi\beta \tag{2.10}$$

All these conditions were obtained above as necessary conditions. Suppose they are satisfied. Then, all that needs to be proved is to establish the second equality of (2.3). From the first equality of (2.4), which is equivalent to the first equality of (2.9), it follows that

$$\xi(t + T) = \xi(t) + \nu\Phi = \xi(t) + 2\pi\alpha + \beta T$$

which, after substituting expression (1.3), reduces to $\alpha\theta(t + T) = \alpha\theta(t) + 2\pi\alpha$. Hence, when $\alpha \neq 0$ we obtain the required equality. If $\alpha = 0$, the second equality of (2.3) easily follows from the quadrature of the second equation of (2.1) by virtue of relations (2.9) and (2.10). Proposition 1 is thereby proved.

The periodicity conditions established here connect the phase and orbital periods by three relations. Nevertheless the phase period (if it exists) must be determined by the solution of system (1.6). This means that, from conditions (2.9) and (2.10) for the periodic solutions, we obtain certain relations for the other parameters of the problem: the factors n and γ and the integration constant.

3. THE FIRST INTEGRALS

From the third equation of (1.4), rewritten in the form

$$[(\alpha W+\beta r)\rho]_\xi+(rV\rho)_r=0$$

it follows that an "invariant stream function" $\psi(\xi, r)$ exists, from which

$$(\alpha W+\beta r)\rho=\psi_r,\quad rV\rho=-\psi_\xi \tag{3.1}$$

The function ψ is the solution of the equation $D\psi = 0$ and keeps a constant value along the integral curves of the equation

$$dr/d\xi=rV/(\alpha W+\beta r) \tag{3.2}$$

by virtue of which these curves are called "invariant streamlines".

Moreover, ψ is a Lagrange coordinate, and hence any solution of the form (1.5) gives a description of the motion of a mass of gas bounded by certain (arbitrary) "invariant streamlines" like impenetrable walls, which can themselves be mobile in the physical plane (r, θ).

Any function $F(\xi, r)$ satisfies the equation $\mathrm{D}F = 0$ if and only if it is a function of ψ, i.e. $F = F(\psi)$. Hence, it is convenient to use the function ψ to find the integrals of system (1.4). Thus, the last equation of (1.4) gives the entropy integral

$$s=s(\psi) \tag{3.3}$$

Further, we will first assume that $\alpha \neq 0$. By multiplying the first of equations (1.4) by αV and the second by $\alpha W + \beta r$ and adding the results, we obtain the energy integral (the analogue of Bernoulli's integral for the steady gas flow

$$\alpha(V^2+W^2)+2\beta rW+2\alpha\frac{\gamma}{\gamma-1}\frac{p}{\rho}=2H(\psi) \tag{3.4}$$

in deriving which we used Eqs (1.1) and (3.3).

The function ψ is found explicitly in the class of solutions (1.5) (we will henceforth assume than $n \neq -2$). Expressions (3.1) take the form

$$\psi_r=r^{n+1}(\alpha B+\beta)R,\quad \psi_\xi=-r^{n+2}AR$$

whence (apart from an unimportant constant term)

$$(n+2)\psi=r^{n+2}(\alpha B+\beta)R \tag{3.5}$$

This enables us to specify $s(\psi)$ and $H(\psi)$ in integrals (3.3) and (3.4). By virtue of representation (1.5), integral (3.3) has the form $r^m S(\xi) = s(\psi)$. Substitution from (3.5)

$$r=[(n+2)\psi/(\alpha B+\beta)R]^{1/(n+2)} \tag{3.6}$$

leads to separation of the variables ξ and ψ, which gives the dependence of the entropy on ψ

$$s(\psi)=s_0\psi^{m/(n+2)}$$

with an arbitrary constant s_0 and a value of the "invariant entropy"

$$S(\xi)=[(\alpha B+\beta)R]^{m/(n+2)} \tag{3.7}$$

Hence, the invariant equation of state takes the form of the relation

$$P=(\alpha B+\beta)^{m/(n+2)}R^{(n+2+2\gamma)/(n+2)} \tag{3.8}$$

A similar procedure with integral (3.4) gives the dependence of the "energy constant" H on ψ

$$H(\psi)=[(n+2)\psi]^{2/(n+2)}$$

while the energy integral (3.4) takes the form of the relation

$$\alpha(A^2+B^2)+2\beta B+2\alpha\frac{\gamma}{\gamma-1}\frac{P}{R}=2h[(\alpha B+\beta)R]^{2/(n+2)} \tag{3.9}$$

with arbitrary constant h.

It turns out that system (1.6) when $\alpha \neq 0$ has one other additional integral, obtained by the following construction. We introduce two new required functions $X(\xi)$ and $Y(\xi)$

$$Y=(\alpha B+\beta)^{-1},\quad X=YP/R \tag{3.10}$$

By virtue of Eq. (3.8) the quantities R, X and Y are connected by the relation

$$Y^{n\gamma}R^{2\gamma}=X^{n+2} \tag{3.11}$$

Hence, the functions B, R and P are expressed explicitly in terms of X and Y. Substituting these expressions into the second and third equations of (1.6) we obtain the equations

$$(\alpha^2\gamma XY-1)X'=\gamma AXY^2[\alpha^2(n+2)X+2\beta] \tag{3.12}$$

$$(\alpha^2\gamma XY-1)Y'=AY^2[\alpha^2(n+2+2\gamma)XY+2\beta Y-2]$$

Hence we obtain the following linear equation for the function $Y = Y(X)$

$$\frac{dY}{dX}=\frac{[\alpha^2(n+2+2\gamma)X+2\beta]Y-2}{\gamma X[\alpha^2(n+2)X+2\beta]} \tag{3.13}$$

the general solution of which also gives the required additional first integral of system (1.6).

Remark 3. For certain values of the factors n and γ the general solution of Eq. (3.13) can be expressed in terms of elementary functions. When $n > 0$ this holds for matched factors

$$n+2=n\gamma \tag{3.14}$$

In this case it follows from Eq. (1.2) that $m = 0$, which, by virtue of relations (1.5) and (3.7), leads to isentropic gas motions.

For matched factors the general solution of Eq. (3.13) is given by the following formulae (with arbitrary constant k)

$$Y=kX^{1/\gamma}(\alpha^2 n\gamma X+2\beta)^{(\gamma-1)/\gamma}+(\alpha n\gamma/2\beta)^2X+1/\beta\quad(\beta\neq 0) \tag{3.15}$$

$$Y=kX+(\alpha^2 n\gamma^2 X)^{-1}\quad(\beta=0)$$

When $\alpha = 0$ ($\beta \neq 0$), integral (3.9) takes the form

$$B=h_1R^{2/(n+2)} \tag{3.16}$$

with a certain constant h_1, while the invariant pressure P, obtained from (3.8), with appropriate normalization, is given by the formula

$$P=RB^{\gamma} \tag{3.17}$$

The additional integral here is derived from the first two equations of (1.6), which take the form

$$\beta A'+A^2-B^2+(n+2)B^{\gamma}=0,\quad \beta B'+2AB=0 \tag{3.18}$$

Hence it follows that the function $A^2(B)$ satisfies a first-order linear equation, the integration of which also gives the required integral (with an arbitrary constant h)

$$A^2 + B^2 - \frac{n+2}{\gamma - 1} B^{\gamma} = hB \tag{3.19}$$

4. KEY EQUATIONS

The first integrals obtained above show that for the complete solution of system (1.6) it is sufficient to obtain the relation between one of the required functions and ξ. This operation reduces to a single quadrature – the solution of a certain key equation.

When $\alpha \neq 0$, we can take as the required function $X(\xi)$, the derivative of which is given by the first equation of (3.12), where $Y = Y(X)$ is defined by an additional integral – the solution of Eq. (3.13) (for matched factors of formula (3.15)), while the function $A = A(X)$ is found from the energy integral (3.9). After substituting expressions (3.8) and (3.10) we rewrite the energy integral in the form

$$A^2 = (\alpha Y)^{-2}\left[\beta^2 Y^2 - 2\left(\alpha^2 \frac{\gamma}{\gamma - 1} X - \alpha h X^{1/\gamma}\right) Y - 1\right] \tag{4.1}$$

When $\alpha \neq 0$ we thereby obtain the key equation

$$\left(\frac{dX}{d\xi}\right)^2 = A^2 \left\{\frac{\gamma X Y^2 [\alpha^2 (n+2) X + 2\beta]}{\alpha^2 \gamma X Y - 1}\right\}^2 \tag{4.2}$$

When $\alpha = 0$, we take as the required function $B(\xi)$, the derivative of which is given by the second equation of (3.18), while the function $A = A(B)$ is found from the additional integral (3.19)

$$A^2 = hB - B^2 + \frac{n+2}{\gamma - 1} B^{\gamma} \tag{4.3}$$

The corresponding key equation is

$$\left(\frac{dB}{d\xi}\right)^2 = A^2 \left\{\frac{2B}{\beta}\right\}^2 \tag{4.4}$$

The specific feature of these equations is the fact that, in the required solutions, the function A^2 is a unique function of X or B, whereas the function $A(X)$ or $A(B)$ will be two-valued.

5. PERIODIC SOLUTIONS

Digressing for the moment from these specific problems, we will point out some general features.

Suppose the function $f(X)$ is continuously differentiable in the open interval $\Delta \subset \mathbb{R}(X)$. Consider the differential equation for the function $X(\xi)$

$$(dX/d\xi)^2 = f(X) \tag{5.1}$$

Proposition 2. If a closed interval $[X_1, X_2] \subset \Delta$ exists such that: (a) $f(X_1) = f(X_2) = 0$, (b) $f(X) > 0$ when $X_1 < X < X_2$ and (c) $f'(X_1) > 0, f'(X_2) < 0$, Eq. (5.1) has a periodic solution $X(\xi)$, even in ξ, with period

$$\Pi = 2\int_{X_1}^{X_2} \frac{dX}{\sqrt{f(X)}} \tag{5.2}$$

In fact, the required solution is constructed as follows: for $0 \leqslant \xi \leqslant \Pi/2$ we take $X'(\xi) = \sqrt{f(X)} > 0$, $X(\xi)$ is defined by the quadrature

$$\xi = \int_{X_1}^{X(\xi)} \frac{dX}{\sqrt{f(X)}}$$

and, by virtue of (5.2), takes the value $X(\Pi/2) = X_2$. For $\Pi/2 \leqslant \xi \leqslant \Pi$ we take $X'(\xi) = -\sqrt{f(X)} < 0$, $X(\xi)$ is defined by the quadrature

$$\xi = \int_{X(\xi)}^{X_2} \frac{dX}{\sqrt{f(X)}} + \frac{\Pi}{2}$$

and, by virtue of (5.2), we take the value $X(\Pi) = X_1$. The solution constructed in the interval $(0, \Pi)$ is continued over the whole axis $\mathbb{R}(\xi)$ as periodic with period Π. It will obviously be even with respect to ξ.

In practice, finding the interval $[X_1, X_2]$, which satisfies conditions (a)–(c), may not be a simple problem. The following sufficient condition, which uses the dependence of the right-hand side in (5.1) on a certain parameter λ, serves this purpose.

Consider the equation

$$(dX/d\xi)^2 = f(X, \lambda) \tag{5.3}$$

and assume that in a certain region $\Omega \subset \mathbb{R}^2(X, \lambda)$ the function $f(X, \lambda)$ is triply continuously differentiable.

Lemma 1. If a point $M(X_0, \lambda_0) \in \Omega$ exists at which

$$1)\, f(M) = 0,\ f_X(M) = 0; \quad 2)\, f_\lambda(M) > 0,\ f_{XX}(M) < 0$$

then, for any sufficiently small ε, the equation

$$(dX/d\xi)^2 = f(X, \lambda_0 + \varepsilon^2) \tag{5.4}$$

has a periodic solution $X_\varepsilon(\xi)$ with period

$$\Pi_\varepsilon = 2\pi/b + O(\varepsilon) \tag{5.5}$$

and, over any finite interval in $\mathbb{R}(\xi)$, the following representation holds

$$X_\varepsilon(\xi) = X_0 - \varepsilon a \cos b\xi + O(\varepsilon^2) \tag{5.6}$$

where the positive constants a and b are given by the equations

$$a^2 = 2f_\lambda(M)/|f_{XX}(M)| \quad 2b^2 = |f_{XX}(M)| \tag{5.7}$$

Proof. The graphs of the functions $f(X, \lambda_0)$ and $f(X, \lambda_0 + \varepsilon^2)$ in the neighbourhood of the point X_0, which result from conditions *1* and *2*, show that for the functions $f(X, \lambda_0 + \varepsilon^2)$ the interval $[X_1, X_2]$, required in Proposition 2, exists. It remains to establish that representations (5.5) and (5.6) hold. Suppose $X = X_0 + x$; it is clear that the differences $X_0 - X_1$ and $X_2 - X_0$ and the quantity x are of the order of ε. By virtue of conditions 1 the expansion of the function $f(X_0, + x, \lambda_0 + \varepsilon^2)$ using Taylor's formula at the point $M(X_0, \lambda_0)$ in the notation of (5.7) has the form

$$f(X_0 + x,\ \lambda_0 + \varepsilon^2) = b^2(\varepsilon^2 a^2 - x^2 + O(\varepsilon^3))$$

Replacement of the variables $(\xi, x) \to (\eta, y)$ using the formulae $\eta = b\xi$, $x = \varepsilon a y$ reduces Eq. (4.4) to $(dy/d\eta)^2 = 1 - y^2 + \varepsilon g$ with a continuous bounded function $g(y, \varepsilon)$, i.e. $|g| + |g_y| < N < \infty$. Hence it follows that the roots of the right-hand side $y_\pm = \pm\sqrt{1 + \varepsilon g(y_\pm, \varepsilon)}$ exist as well as the definition of the relation $y = y(\eta)$ in terms of the quadrature

$$\eta = \int_{y_-}^{y} \frac{ds}{\sqrt{1 - s^2 + \varepsilon g(s, \varepsilon)}} \tag{5.8}$$

Finally, representations (5.5) and (5.6) are obtained by bilateral estimates of the integral based on the equality $|g| + |g_y| < N$.

The following fact is useful for analysing the solutions of an equation of the form (5.3), which satisfy the conditions of Lemma 1. Certain quantities (relations between quantities) remain interesting in the limit as $\varepsilon \to 0$. These quantities (relations) will be called *limit* quantities (relations).

For example, for solutions of the form (5.6) the limit quantities are X_0 and λ_0, and also the limit period $\Pi_0 = 2\pi/b$, which follows from (5.5). The limit relations give a good reference point when constructing accurate solutions (usually by numerical calculation), and also for a clear qualitative representation of the periodic motions described. In addition, limit relations are useful for establishing approximate relations between other parameters, on which the function f may depend.

Remark 4. For periodic solutions of system (1.6) it follows from Proposition 1 that if B_0 is the limit value of the function $B(\xi)$, then the limit forms of relations (2.9) and (2.10) lead to expressions for the limit periods

$$\alpha(T_0 B_0 - 2\pi) = 0, \quad \beta T_0 = 2\pi(\nu/\mu - \alpha), \quad \Phi_0 = 2\pi/\mu \tag{5.9}$$

Moreover, if a quantity b (5.7) is defined for solving the corresponding key equation (4.2) or (4.4) and this solution is periodic with period Φ, its limit value may be identical with Π_0, i.e. we should have $\Phi_0 = 2\pi/b$. The following limit relation therefore follows from Eqs (5.9)

$$b = \mu \tag{5.10}$$

6. THE GAS PENDULUM

The "standard" solution with values $(\alpha, \beta) = (0, 1)$ was called a "gas pendulum" in [2]. However, the limit relations were not derived there; this gap is filled here.

In key equation (5.4) for this case $X = B$, $\xi = t$, $\lambda = h$, while the function f is such that

$$f(B, h) = 4B^3\left(h - B + \frac{n+2}{\gamma - 1}B^{\gamma-1}\right) \tag{6.1}$$

The region Ω is the half-plane $B > 0$, $h \in \mathbb{R}$; we will assume that $n + 2 \neq 0$ and $\gamma > 1$. Equations *1* from the condition of Lemma 1 are easily solved, and the point $M(B_0, h_0)$ is found from the equations

$$B_0^{2-\gamma} = n + 2, \quad (\gamma - 1)h_0 = (\gamma - 2)B_0 \tag{6.2}$$

The values of the derivatives

$$f_h(M) = 4B_0^3, \quad f_{BB}(M) = 4(\gamma - 2)B_0^2$$

obtained when calculating (6.2) show that conditions 2 of Lemma 1 are satisfied if and only if $\gamma > 2$. In this case the constants a and b in (5.7) have the values

$$a = \sqrt{2B_0/(2-\gamma)}, \quad b = B_0\sqrt{4 - 2\gamma} \tag{6.3}$$

The periodicity conditions (2.9) and (2.10) here are

$$\nu\int_0^\Phi B(t)dt = 2\pi, \quad \nu\Phi = \mathrm{T}, \quad \mu\Phi = 2\pi \tag{6.4}$$

By Lemma 1, Eq. (4.4) with right-hand side $f(B, h_0 + \varepsilon^2)$ has a Φ-periodic solution

$$B(t) = B_0 - \varepsilon a \cos bt + O(\varepsilon^2) \tag{6.5}$$

By substituting solution (6.5) into the first relation of (6.4) it can be shown that for sufficiently small ε all the relations of (6.4) are satisfied only if $\nu = \sqrt{4 - 2\gamma}$ and $B_0 = \mu$. Of course we must have $\nu = 1$

and $\gamma = \frac{3}{2}$. In this case the phase and orbital periods are identical: $\Phi = T = 2\pi/\mu$. Limit relations (5.9) and (5.10) also give the same result.

In the interpretation of the motion described as a periodic compression and expansion of a rotating gas cylinder [2] this means that after one cycle (period T) the gas completely recovers the initial state of motion, i.e. one obtains a "gas pendulum", pulsing with frequency μ.

7. FLOWS WITH CLOSED STREAMLINES

In the second "standard" case $(\alpha, \beta) = (1, 0)$ we will have $\xi = \theta$, which leads to steady gas flows. A similar solution was considered in [3] with the participation there of the self-similarity factor α, which is equal to unity in solutions of the form (1.5). In addition, an analysis was carried out in [3] for isentropic flows, i.e. for $m = 0$ (Remark 1). Here this result is supplemented by indicating the limit forms of this kind of gas flow, in particular, by determining its periods.

The key equation (4.2) in this case takes the form

$$(dX/d\theta)^2 = A^2[n\gamma^2X^2Y^2/(\gamma XY-1)]^2 \tag{7.1}$$

Expression (3.15) for Y in terms of X, after replacing the constant $k \to k/n\gamma^2$ is given by the relation

$$n\gamma^2XY = kX^2+1 \tag{7.2}$$

while the dependence of A^2 on X is determined by the energy integral (3.9)

$$Y^2A^2 = 2hX^{1/\gamma}Y - n\gamma XY - 1 \tag{7.3}$$

Here the parameter $\lambda = h$. The region Ω, in which the right-hand side $f(X, h)$ of Eq. (7.1) satisfies the smoothness condition from Lemma 1, is fixed by the inequalities

$$X>0, \quad Y>0; \quad \gamma XY>1 \tag{7.4}$$

Conditions 1 of Lemma *1* lead to an equation for the quantity $Z = kX^2$

$$(\gamma-1)Z^2-(\gamma^2-\gamma+2)Z+\gamma^2-1=0$$

with roots $Z_1 = \gamma-1$, $Z_2 = (\gamma+1)/(\gamma-1)$. But it follows from relation (7.2) that $\gamma XY = (Z+1)(\gamma-1)/2\gamma$, and with the root Z_1 it will be $\gamma XY = (\gamma-1)/2$ while with the root Z_2 it will be $\gamma XY = 1$. Hence only the root Z_1 applies, from which the point $M(X_0, h_0)$ is uniquely defined (h_0 is found from the condition for the right-hand side of (7.3) to be equal to zero). By inequalities (7.4), the membership $M \in \Omega$ is only possible if $\gamma > 3$.

Calculation of the derivative f_{xx} of the right-hand side of $F(X, h)$ of Eq. (7.1) leads to the quantity

$$f_{XX}(M) = -8(\gamma-1)/(\gamma-3) \tag{7.5}$$

Of course, when $\gamma > 3$ conditions 2 of the lemma are satisfied. Formula (5.7) defines the quantity $b^2 = 4(\gamma-1)/(\gamma-3)$. From the limit relation (5.10) we find possible factors $\gamma = (3\mu^2-4)/(\mu^2-4)$. The greatest value of γ from which periodic gas motion of the form (1.5) exists, is obtained when $\mu = 3$ and is equal to 23/5.

The periodicity conditions (2.9) give the equation $\mu = \nu$. According to definition (3.10), here $B = 1/Y$. Hence, the limit value of the orbital period, obtained from (5.9), is equal to $T_0 = 2\pi Y_0$, where Y_0 is found from relation (7.2), and we finally obtain

$$T_0 = \pi\sqrt{(\gamma-1)k}/\gamma \tag{7.6}$$

where the constant $k > 0$ remains arbitrary.

The closed streamlines of this gas flow are described by formula (3.6). They form an "asterisk" in the (r, θ) plane with ν-bulges and hollows, and particles moving along these lines return to the initial position after a time interval (7.6).

8. THE GAS OPINION

A new form of periodic gas motion is described by the "standard" solution with $(\alpha, \beta) = (1, 1)$, when $\xi = \theta + t$. To simplify the formulae, we will here consider the isentropic version, when $m = 0$ (Remark 1), with $\gamma > 1$ and $n > 0$.

The key equation (4.2) in this case takes the form

$$(dX/d\xi)^2 = f(X,h) \equiv G^2(X)F(X,h) \tag{8.1}$$

$$G(X) = \frac{\gamma XY(n\gamma X+2)}{\gamma XY-1}, \quad F(X,h) = Y^2 - n\gamma XY - 1 + 2hX^{1/\gamma}Y$$

$$Y = kX^{1/\gamma}(n\gamma X+2)^{(\gamma-1)/\gamma} + (n\gamma/2)^2 X + 1$$

We will take $\lambda = h$ as the parameter, while the region Ω of smoothness of the right-hand side of (8.1) is specified by the inequalities

$$X > 0, \quad Y > 0, \quad \gamma XY \neq 1$$

Conditions *1* of Lemma 1 reduce to the equations $F = 0$ and $F_X = 0$ and define the coordinates of the points $M(X_0, h_0) \in \Omega$ in terms of the quantity Y_0 from the relations

$$n\gamma X_0 Y_0 = (Y_0 - 1)^2, \quad h_0 X_0^{1/\gamma} Y_0 = 1 - Y_0 \tag{8.2}$$

The first of conditions *2* of Lemma 1 is satisfied since $f_h = 2G^2(X_0)X_0^{1/\gamma}Y_0 > 0$ in Ω. To realise the second condition we calculate the derivative f_{XX}, the value of which at the point $M(X_0, h_0)$ (obtained by quite long calculations) is

$$f_{XX}(M) = -4(Y_0 - 1)^2\varphi(Y_0,n), \quad \varphi(Y_0,n) = (2 - nY_0)/[(Y_0-1)^2 - n] \tag{8.3}$$

Hence, the second condition 2 of Lemma 1 is equivalent to the inequalities $Y_0 \neq 1$ and $\varphi(Y_0, n) > 0$. The latter is satisfied if and only if the point (Y_0, n) belongs to the union of D with the region $D_i \subset \mathbb{R}^2(Y_0, n)$:

$$D_1 = \{n < 1, \ 0 < Y_0 < 1 - \sqrt{n}\}$$

$$D_2 = \{n < 1, \ 1 + \sqrt{n} < Y_0 < 2/n\}$$

$$D_3 = \{n > 1, \ 2/n < Y_0 < 1 + \sqrt{n}\}$$

Formula (8.3) gives $\varphi(Y_0, 1) = -1/Y_0$, i.e. the points $\varphi(Y_0, 1) \notin D$; hence necessarily $n \neq 1$ $(\gamma \neq 3)$.

By virtue of Lemma 1, periodic solutions of Eq. (8.1) exist. The quantity b, according to (5.7), is given by the equation

$$b^2 = 2(Y_0-1)^2\varphi(Y_0, n) \tag{8.4}$$

By relations (3.10), $B_0 = (1-Y_0)/Y_0$. The expression $Y_0 = 1-\mu/\nu$ follows from (5.9), and for the limit periods we obtain the formulae

$$T_0 = 2\pi(\nu-\mu)/\mu, \quad \Phi_0 = 2\pi/\mu \tag{8.5}$$

By virtue of relation (5.10) Eq. (8.4) reduces to a relation which defines the possible factors n

$$n = \nu(\mu^2-4)/(\nu^3-2\nu+2\mu) \tag{8.6}$$

The results obtained reduce to the following formulation.

Theorem. For any integer $\mu \neq 0$ and $\nu > 0$, with which formula (8.6) gives the quantity $n > 0, n \neq 1$, the system of equations (1.6) with $(\alpha, \beta) = (1, 1)$ with values of n (8.6) and $\gamma = (n + 2)/n$ has a solution describing the periodic gas motion with limit periods (8.5).

Proof. The relations indicated in the condition of the theorem guarantee that the point (Y_0, n) falls in the region D, where $Y_0 = 1-\mu/\nu$. The rest follows from Lemma 1 and Note 4.

The equation $r = r(\xi)$ of the "invariant streamlines" $L_\psi(\psi = \text{const})$, by relations (3.6), (3.10) and (3.1), here reduces to the form

$$r = r_0 X^{-1/\gamma} Y^{1/2} \tag{8.7}$$

where r_0 depends only on ψ, while the functions $X = X(\xi)$, $Y = Y(\xi)$ are periodic with phase period Φ. The lines L_ψ are closed in the plane of the polar coordinate, (r, ξ), i.e. they form an "asterisk" with $|\mu|$ "teeth". Unlike the situation described in Section 7 here the "asterisk" (8.7) in the (r, θ) plane rotates with limit angular velocity $d\xi/dt = 1/Y_0$, while the gas particles run round the line L_ψ and return to the initial position after a time T. This form of the gas motion has been called the "gas pinion", the number of "teeth" of which can be any integer $|\mu| \neq 2$. It is curious that when $Y_0 > 1$ the motion of the gas particles occurs in a direction opposite to the rotation of the pinion.

We note in conclusion that the limitations of a journal article has not enabled us to describe all the results obtained in more detail, in particular, the existence of a non-isentropic "gas pinion". We can only raise the question of whether an exhaustive description of all possible forms of periodic gas motions when there are no external forces or energy actions has been given by these investigations. Although there are some "general considerations" which favour a positive answer, the question still remains open.

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